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# Dynamical affine symmetry and renormalisation in gravity

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**Abstract.** The Einstein Lagrangian is written in terms of the Cartan forms of the affine group. An each-order-covariant renormalisation procedure is formulated within the external background field method. The approach proposed is proved to be equivalent on the mass shell to the ordinary one.

## 1. Introduction

Formally, the theory of quantum gravity is non-renormalisable. Nevertheless, the study of ultraviolet divergence is of certain interest and may expose a non-trivial link between the symmetry of the theory and its renormalisability. Brown (1973), Capper *et al* (1973) and 't Hooft and Veltman (1974) have shown that the sum of counterterms of the one-loop approximation equals zero on the mass shell. Recently Kallosh *et al* (1977) have also proved the same result off the mass shell. It is therefore interesting to look into higher orders of perturbation theory.

In this paper, we propose to study the ultraviolet divergences in gravity by using the dynamical affine symmetry.

It is well known that the theory of gravity is a gauge field theory (Feynman 1963, De Witt 1967, Faddeev and Popov 1967, 1973), but here we use the analogy with chiral dynamics. As Borisov and Ogievetsky (1974) have shown, gravitational field theory is a theory of spontaneous breakdown of affine and conformal symmetries in the same way as chiral dynamics is a theory of spontaneous breakdown of chiral symmetry.

A method of renormalisation based on the Lagrangian symmetry has been formulated for chiral theories by Kazakov *et al* (1977a, b) and consists of the following.

(1) The each-order-invariant perturbation theory is constructed by using the external background field method. For theories with nonlinear realisation of symmetry one should use the group summation $\phi\,(+)\,h$ in the change $\phi + h$.

(2) The sum of counterterms in each perturbation order is written as an expansion over the complete set of invariants of a group (of Lagrangian symmetry) of appropriate dimension. The invariants (including the Lagrangian) are constructed and classified over perturbation orders with the help of Cartan forms (Cartan 1946, Volkov 1973).

(3) To obtain numerical coefficients for the group invariants, the singularities of the minimum number of loop diagrams must be calculated.

In this paper, we apply the above renormalisation programme (1)–(3) to gravity. The paper is organised as follows. In §§ 2 and 3 each-order-invariant perturbation theory is formulated within the external background field method. In § 4 the equivalence on the mass shell is established between a modified generating functional of loop diagrams introduced in §§ 2 and 3 and a standard one (Faddeev 1973). In § 5 we construct the gravitational Lagrangian and complete set of invariants in terms of the Cartan forms of the group of affine transformations. The structure equations of the affine group are also deduced and exploited. It is shown that the covariant derivative can be determined by requiring, instead of the conformal symmetry (Borisov and Ogievetsky 1974), that the metric tensor covariant derivative vanishes identically. At the same time, a more formal analogy with chiral dynamics is achieved, which is important because of the absence of the conformal-invariant regularisation (Duff 1977).

## 2. Perturbation theory

For loop diagrams the background field method yields the generating functional

$$Z(\phi)=N^{-1}\int\prod_{\nu\leqslant\mu,x}\mathrm{d}h_{\mu\nu}(x)\prod_{x}\delta(f(\phi+h))$$
$$\times\Delta_f(\phi+h)\exp\left[\mathrm{i}\int\mathrm{d}^4x\left(\mathscr{L}(\phi+h)-\frac{\delta\mathscr{L}(\phi)}{\delta\phi_{\alpha\beta}}h_{\alpha\beta}-\mathscr{L}(\phi)\right)\right] \tag{1}$$

where the $\phi_{\mu\nu}$ are external (background) fields and the $h_{\mu\nu}$ are internal (quantum) fields, $N$ is a norm, $f(\phi+h)=0$ is a gauge-fixing equation, $\Delta_f$ is the Faddeev–Popov determinant and $\mathscr{L}$ the Einstein Lagrangian.

The generating functional (1) is equivalent, up to tree graphs, to the standard generating functional for Green functions ('t Hooft 1973, Kallosh 1974, Grisaru and van Niewenhuizen 1975)

$$Z(J)=N^{-1}\int\prod_{\mu\leqslant\nu,x}\mathrm{d}h_{\mu\nu}(x)\prod_{x}\delta(f(h))\Delta_f(h)\exp\left(\mathrm{i}\int\mathrm{d}^4x(\mathscr{L}(h)+J^{\alpha\beta}h_{\alpha\beta})\right) \tag{2}$$

provided the external fields $\phi_{\mu\nu}$ obey the equations

$$\delta\mathscr{L}(\phi)/\delta\phi_{\alpha\beta}=-J^{\alpha\beta} \tag{3}$$

with $J^{\alpha\beta}$ an external source. The relation (3) is a formal relation between functional variables $\phi_{\alpha\beta}$ and $J^{\alpha\beta}$.

Grisaru and van Niewenhuizen (1975) and Kallosh (1974) have shown that for pure gravity and a pure Yang–Mills field the counterterms resulting from (1) are gauge independent (under certain conditions on $\delta(f(h))$).

For studying ultraviolet divergence, it is sufficient to find the Lagrangian counterterms. These are obtained by expanding the integrand (1) in the series in $h_{\alpha\beta}$. For a theory of the linear realisation of symmetry a series of this type is each-order invariant. For the nonlinear, i.e. dynamical, realisation of symmetry by the Goldstone fields, the each-order invariance of the series in $h_{\alpha\beta}$ and, consequently, the invariance of perturbation theory in each order in the number of loops are broken. The reason is that in this case the field space is curved, and the operation $\delta^n/\delta h^n_{\alpha\beta}$ is no longer covariant. The each-order-invariant series can be constructed by using the covariant variational derivative $\mathrm{D}^n/\mathrm{D}^n_{\alpha\beta}$ (Honerkamp 1972) instead.

We shall denote the Taylor series with covariant derivatives using the symbol (+):

$$f(\phi(+)h)=f(\phi)+\sum_{n=1}^{\infty}\frac{1}{n!}\frac{\mathrm{D}^n f(\phi)}{\mathrm{D}\phi^n}h^n. \tag{4}$$

Such an expansion corresponds to the transition from point $\phi$ to point $\phi+h$ in the field space along geodesics. Also, the relation (Honerkamp 1972)

$$f(\phi(+)h)=f(\xi(\tau))|_{\tau=0}+\sum_{n=1}^{\infty}\frac{1}{n!}\frac{\delta^n f(\xi(\tau))}{\delta\tau^n}\bigg|_{\tau=0} \tag{5}$$

is valid; here $\xi(0)=\phi$, $\xi(1)=\phi+h$ and $\xi(\tau)$ satisfies the geodesic equation in the field space

$$\frac{\mathrm{d}^2\xi^\mu}{\mathrm{d}\tau^2}+\Gamma^\mu_{\nu\rho}\frac{\mathrm{d}\xi^\nu}{\mathrm{d}\tau}\frac{\mathrm{d}\xi^\rho}{\mathrm{d}\tau}=0.$$

Here $\Gamma^\mu_{\nu\rho}$ is the Christoffel symbol in the field space, indices $\mu$, $\nu$, $\rho$ run from 1 to $N$ where $N$ is the dimension of the Goldstone field space (for gravity $N=10$). The right-hand sides of (4) and (5) coincide in each order (Honerkamp 1972). Therefore, to construct the each-order-invariant series (4) requires a parametrisation $f(\xi(\tau))$ with the given properties. The latter problem is solved by using the Cartan forms.

## 3. Cartan forms, group summation

Let $G$ be a $(k+r)$ parametric semi-simple group (of Lagrangian symmetry) which gives rise to vacuum degeneration and production generation of Goldstone particles and let $H$ be its maximum subgroup leaving the vacuum invariant.

Cartan forms $\omega$ and $\theta$ are defined by the finite transformations of group $G$ through the equality

$$G^{-1}(h)\partial_\mu G(h)=\mathrm{i}(\omega^m_\mu(h)X_m+\theta^n_\mu(h)Y_n) \tag{6}$$

with the initial conditions

$$\omega^m_\mu(0)=0 \qquad \theta^n_\mu(0)=0$$

where $h$ are the group parameters, $Y_n(n=1,\ldots,r)$ are the generators of subgroup $H$ and $X_m(m=1,\ldots,k)$ are the generators of the factor space $G/H$. The form $\theta$ is used to construct the covariant derivative and the forms $\omega$ and their covariant derivatives are used to construct the complete set of invariants of group $G$, in particular the Lagrangian (Volkov 1973). The complete set of group invariants looks like

$$[\mathrm{D}^{n_1}\omega]^{n_2}[\omega]^{n_3}$$

with $n_1$, $n_2$, $n_3$ positive integers, all the indices being contracted.

On the other hand, it is known (Cartan 1946, Volkov 1973) that the form $\omega(h)$ describes the transition from the zero point to point $h$ in the parameter space. In particular, in the exponential parametrisation of group elements $G=\exp\{\mathrm{i}h^m X_m\}$ the form $\omega(h)$ describes the transition along a geodesic.

Making the change

$$G(h)\to G(\phi)G(h) \tag{7}$$

in (6) and solving the equations for the Cartan forms with nonzero boundary conditions (Pervushin 1975, 1976)

$$\begin{aligned}\bar{\omega}(\phi, 0) &= \omega(\phi), \\ \bar{\theta}(\phi, 0) &= \theta(\phi),\end{aligned} \tag{8}$$

we arrive at the extended Cartan forms $\bar{\omega}(\phi, h)$ and $\bar{\theta}(\phi, h)$. The form $\bar{\omega}(\phi, h)$ describes the transition from point $\phi$ to point $\phi+h$ in the field space, and in the particular case of the exponential parametrisation, the transition along a geodesic. The changes (7) and (8) are called the summation in the factor space.

Once the parameters of the factor space are identified with fields of Goldstone particles (Volkov 1973), the summation in the factor space coincides with the operation (+) defined by (4).

To construct the each-order-invariant perturbation theory, group invariants, and in particular the Lagrangian, should be expressed in terms of the Cartan forms of the factor space and the relation

$$\mathscr{L}(\phi\,(+)\,h) = \mathscr{L}(\bar{\omega}(\phi, h))$$

is to be applied.

Then the generating functional of the each-order-invariant perturbation theory is of the form

$$\begin{aligned}Z(\phi) = N^{-1} \int \prod_{\nu \leqslant \mu, x} \mathrm{d}h_{\mu\nu}(x) \prod_{x} \delta(f(\phi\,(+)\,h)) \\ \times \Delta_f(\phi\,(+)\,h) \exp\left[\mathrm{i} \int \mathrm{d}^r x \left(\mathscr{L}(\phi\,(+)\,h) - \frac{\mathrm{D}\mathscr{L}(\phi)}{\mathrm{D}\phi_{\alpha\beta}} h_{\alpha\beta} - \mathscr{L}(\phi)\right)\right].\end{aligned} \tag{9}$$

Before calculating $\mathscr{L}(\bar{\omega}(\phi, h))$ we shall prove the equivalence on the mass shell between the generating functional (9) and the standard expression (2) up to tree graphs.

## 4. Equivalence theorem

Change the variables

$$h \to f(\phi, h) \equiv \phi\,(+)\,h \tag{10}$$

in the functional (2). Using the fact that the determinant of (10) is unity (Borisov and Ogievetsky 1974), and the equality $J \times (\phi\,(+)\,h) = J \times \phi + J \times h$ on the mass shell (Kallosh 1974), we then derive the formula

$$\begin{aligned}Z(J, \phi) = N^{-1} \exp\left(\mathrm{i} \int \mathrm{d}^4 x\, J(x) \phi(x)\right) \int \prod_{\nu \leqslant \mu, x} \mathrm{d}h_{\mu\nu}(x) \prod_{x} \delta(f(\phi\,(+)\,h)) \Delta_f(\phi\,(+)\,h) \\ \times \exp\left[\mathrm{i} \int \mathrm{d}^4 x \left(\mathscr{L}(\phi) + \frac{\mathrm{D}\mathscr{L}(\phi)}{\mathrm{D}\phi} h + \frac{1}{2} \frac{\mathrm{D}^2\mathscr{L}(\phi)}{\mathrm{D}\phi^2} h^2 + \ldots + J \times h\right)\right].\end{aligned} \tag{11}$$

Provided equation (3) holds and the first-order covariant derivative of the scalar coincides with the ordinary one, the terms $(\mathrm{D}\mathscr{L}(\phi)/\mathrm{D}\phi)h$ and $J \times h$ in (11) cancel and up to tree graphs we obtain (9).

## 5. Gravity in terms of Cartan forms of the group of affine transformations

It is known that the gravitational field is a gauge field which provides the invariance of the Einstein theory under the general coordinate transformations. It is just a deep analogy between the gravitational field and Yang–Mills vector fields, gauge fields for internal symmetries.

Another deep analogy also exists between gravitons in gravity theory and pions in chiral dynamics—SU(2) × SU(2) based on nonlinear realisations of chiral symmetry (Borisov and Ogievetsky 1974).

The algebra of the affine group $A(4) = P_4 \otimes L(4, R)$ consists of generators of the Lorentz group $L_{\mu\nu}$, generators of proper affine transformations $R_{\mu\nu}$ and those of translation $P_\mu$:

$$\frac{1}{\mathrm{i}}\left[L_{\mu\nu}, L_{\rho\tau}\right] = \delta_{\mu\rho}L_{\nu\tau} - \delta_{\mu\tau}L_{\nu\rho} - (\mu \leftrightarrow \nu)$$

$$\frac{1}{\mathrm{i}}[L_{\mu\nu}, R_{\rho\tau}] = \delta_{\mu\rho}R_{\nu\tau} + \delta_{\mu\tau}R_{\nu\rho} - (\mu \leftrightarrow \nu)$$

$$\frac{1}{\mathrm{i}}[R_{\mu\nu}, R_{\rho\tau}] = \delta_{\mu\rho}L_{\tau\nu} + \delta_{\mu\tau}L_{\rho\nu} + (\mu \leftrightarrow \nu)$$

$$\frac{1}{\mathrm{i}}[L_{\mu\nu}, P_\rho] = \delta_{\mu\rho}P_\nu - \delta_{\nu\rho}P_\mu$$

$$\frac{1}{\mathrm{i}}[R_{\mu\nu}, P_\rho] = \delta_{\mu\rho}P_\nu + \delta_{\nu\rho}P_\mu.$$

Consider nonlinear transformations in the factor space $A/L$ whose parameters are the coordinates $x_\mu$ and ten Goldstone fields, $h_{\mu\nu}$, gravitons. Invariants under linear transformations with constant parameters are constructed with the help of Cartan forms:

$$\begin{aligned} G^{-1}\, dG &= \mathrm{i}[\omega^{P}_{\mu}(d)P_\mu + \tfrac{1}{2}\omega^{R}_{\mu\nu}(d)R_{\mu\nu} + \tfrac{1}{2}\omega^{L}_{\mu\nu}(d)L_{\mu\nu}] \\ G &= \exp\{\mathrm{i}P_\mu x_\mu\}\exp\{\tfrac{1}{2}\mathrm{i}h_{\alpha\beta}R_{\alpha\beta}\}. \end{aligned} \tag{12}$$

The form $\omega^R$ defines the covariant differential for Goldstone fields $h$, and the forms $\omega^P$ and $\omega^L$ the covariant differentiation of external fields $\psi$ transformed by representations of the Lorentz group with generators $L^{\psi}_{\mu\nu}$:

$$\begin{aligned} \nabla_\lambda \psi &= \mathrm{D}\psi/\omega^{P}_{\lambda} \\ \mathrm{D}\psi &= (d + \tfrac{1}{2}\mathrm{i}\omega^{L}_{\mu\nu}(d)L^{\psi}_{\mu\nu})\psi. \end{aligned} \tag{13}$$

Cartan forms in the exponential parametrisation which corresponds to the choice of normal coordinates in the ten-dimensional space $h_{\mu\nu}$ are (Borisov and Ogievetsky 1974):

$$\begin{aligned} &\omega^{P}_{\mu}(d) = r_{\mu\nu}\,\mathrm{d}x_\nu \qquad \omega_{\mu\nu}(d) = r^{-1}_{\mu\sigma}\,\mathrm{d}r_{\sigma\nu} \\ &\omega^{R}_{\mu\nu}(d) = \tfrac{1}{2}(\omega_{\mu\nu}(d) + \omega_{\nu\mu}(d)) \equiv \omega_{(\mu\nu)} \\ &\omega^{L}_{\mu\nu}(d) = \tfrac{1}{2}(\omega_{\mu\nu}(d) - \omega_{\nu\mu}(d)) \equiv \omega_{[\mu\nu]} \\ &r_{\mu\nu} = (\mathrm{e}^{h})_{\mu\nu} \qquad r^{-1}_{\mu\nu} = (\mathrm{e}^{-h})_{\mu\nu}. \end{aligned} \tag{14}$$

The invariant elements of length and volume are constructed from the forms $\omega^P$:

$$(\mathrm{d}s)^2 = \omega^P_\lambda \omega^P_\lambda = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu \qquad g_{\mu\nu} = r_{\mu\sigma} r_{\sigma\nu}$$
$$\mathrm{d}v = \sqrt{-g}\,\mathrm{d}^4x = -\mathrm{i}\omega^P_1(d)\omega^P_2(d)\omega^P_3(d)\omega^P_4(d). \tag{15}$$

The requirement of the minimum with respect to the number of derivatives does not fix the theory uniquely because the transformation properties of the covariant derivative (13) do not change if one adds several terms of the same order of derivative with arbitrary coefficients $c_1, c_2, c_3$:

$$\bar{\nabla}_\lambda \psi = \eth_\lambda \psi + \tfrac{1}{2}\mathrm{i} v_{\mu\nu,\lambda}(c_1, c_2, c_3) L^\psi_{\mu\nu}\psi \qquad \eth_\lambda = r^{-1}_{\lambda\gamma}\,\partial_\gamma$$
$$v_{\mu\nu,\lambda} = \omega_{[\mu\nu],\lambda} + c_1[\omega_{(\nu\lambda),\mu} - \omega_{(\mu\lambda),\nu}] + c_2[\delta_{\mu\lambda}\omega_{(\sigma\sigma),\nu} - \delta_{\nu\lambda}\omega_{(\sigma\sigma),\mu}] \tag{16}$$
$$+ c_3[\delta_{\mu\lambda}\omega_{(\nu\sigma),\sigma} - \delta_{\nu\lambda}\omega_{(\mu\sigma),\sigma}] \qquad \omega_{\mu\nu,\gamma} = r^{-1}_{\mu\sigma}\eth_\gamma r_{\sigma\nu}.$$

As has been shown by Borisov and Ogievetsky (1974), the parameters $c_1, c_2, c_3$ are defined uniquely by the requirement that the theory be at the same time the nonlinear realisation of the conformal group

$$c_1 = -1 \qquad c_2 = 0 \qquad c_3 = 0. \tag{17}$$

This requirement leads to the tensor field theory whose equations coincide with the Einstein ones.

In this paper we want to indicate that the ambiguity of the theory of nonlinear affine realisation may be removed naturally by requiring the covariant derivative of the metric tensor to be identically zero:

$$\bar{\nabla}_\lambda g_{\mu\nu} \equiv 0 \tag{18}$$

where $\bar{\nabla}_\lambda$ is determined by the relations (16).

A covariant form for the Goldstone fields $h_{\mu\nu}$ may be found by considering the commutator of covariant derivatives for any field $\psi$:

$$(\bar{\nabla}_\lambda \bar{\nabla}_\rho - \bar{\nabla}_\rho \bar{\nabla}_\lambda)\psi = \tfrac{1}{2}\mathrm{i} R_{\mu\nu,\lambda\rho} L^\psi_{\mu\nu}\psi$$

where

$$R_{\mu\nu,\lambda\rho} = \eth_\lambda v_{\mu\nu,\rho} + v_{\mu\nu,\gamma} v_{\rho\gamma,\lambda} + v_{\mu\gamma,\rho} v_{\nu\gamma,\lambda} - (\lambda \leftrightarrow \rho). \tag{19}$$

The Riemann tensor $R_{\mu\nu\lambda\rho}$, the Ricci tensor $R_{\mu\lambda}$ and the scalar curvature $R$ are connected with the quantity $R_{\mu\nu,\lambda\rho}$ in the following way (Borisov and Ogievetsky 1974):

$$R_{\mu\nu\lambda\rho} = r_{\mu\bar{\mu}} r_{\nu\bar{\nu}} r_{\lambda\bar{\lambda}} r_{\rho\bar{\rho}} R_{\bar{\mu}\bar{\nu},\bar{\lambda}\bar{\rho}}$$
$$R_{\mu\lambda} = r_{\mu\bar{\nu}} r_{\lambda\bar{\lambda}} R_{\bar{\mu}\nu,\bar{\lambda}\nu}$$
$$R = R_{\mu\nu,\mu\nu}.$$

The quantity $R_{\mu\nu\lambda\rho}$ and its contractions are expressed in terms of Cartan forms of both the factor space $\omega^R$ and of the vacuum stability subgroup $\omega^L$. In order to expand the gravitational Lagrangian $R\sqrt{-g}$ in a Taylor series one should, as revealed in § 3, express it in terms of Cartan forms (and its covariant derivatives) of just the factor space $A/L$.

The general form of an invariant of the affine group is

$$[\nabla^{n_1}\omega^R]^{n_2}[\omega^R]^{n_3}[\nabla^{n_4}\omega^P]^{n_5}[\omega^P]^{n_6} \tag{20}$$

where $\nabla$ is the affine covariant derivative (13) and $n_1, \ldots, n_6$ are positive integer numbers. In the case when not all the indices are contracted, the corresponding expression is covariant (i.e. it is transformed by representations of the affine group.)

It follows from dimensional considerations that $R_{\mu\nu,\lambda\rho}$ may be written in the form

$$R_{\mu\nu,\lambda\rho} = (c_\alpha[\nabla\omega^R]_\alpha + c_\beta[\omega^R]^2_\beta)_{\mu\nu,\lambda\rho}. \tag{21}$$

Here $c_\alpha$, $c_\beta$ are the numerical coefficients which have to be determined. Indices $\alpha$, $\beta$ mean that several terms of the $\nabla\omega^R$ and $[\omega^R]^2$ type exist (different ways of contraction).

Let us show that due to the Cartan structural equations (Cartan 1946), all unwanted terms in (19) really do disappear and $R_{\mu\nu,\lambda\rho}$ may be written in the form (21).

### 5.1. *Affine group structural equations*

Structural equations express the fact that two infinitesimal group elements commute when acting upon the scalar (of the affine group). In our case we have

$$(\delta_\lambda\delta_\rho - \delta_\rho\delta_\lambda)f = 0 \tag{22}$$

$$\delta_\xi = \mathrm{i}[\omega^P_{\mu,\xi}P_\mu + \tfrac{1}{2}\omega^R_{\mu\nu,\xi}R_{\mu\nu} + \tfrac{1}{2}\omega^L_{\mu\nu,\xi}L_{\mu\nu}]. \tag{23}$$

Affine group structural equations will be obtained after the cumbersome transformations of substituting (23) into (22):

$$\begin{aligned}
(\omega^P_\mu)'_{\lambda\rho} &= [\omega^P_\xi, \omega^R_{\xi\mu}]_{\lambda\rho} + [\omega^P_\xi, \omega^L_{\xi\mu}]_{\lambda\rho} \\
(\omega^R_{\mu\nu})'_{\lambda\rho} &= 2[\omega^R_{\xi\nu}, \omega^L_{\xi\mu}]_{\lambda\rho} \\
(\omega^L_{\mu\nu})'_{\lambda\rho} &= [\omega^R_{\xi\mu}, \omega^R_{\xi\nu}]_{\lambda\rho} + [\omega^L_{\xi\nu}, \omega^L_{\xi\mu}]_{\lambda\rho}
\end{aligned} \tag{24}$$

where

$$(\omega^L_{\mu\nu})'_{\lambda\rho} = \delta_\lambda\omega^L_{\mu\nu}(\eth_\rho) - \delta_\rho\omega^L_{\mu\nu}(\eth_\lambda)$$

$$[\omega^L_{\xi\nu}, \omega^L_{\xi\mu}]_{\lambda\rho} = \omega^L_{\xi\nu,\rho}\omega^L_{\xi\mu,\lambda} - \omega^L_{\xi\nu,\lambda}\omega^L_{\xi\mu,\rho};$$

the same holds for $\omega^P$ and $\omega^R$.

By substituting relation (24) into relation (19) we obtain

$$\begin{aligned}
R_{\mu\nu,\lambda\rho} = {}& \nabla_\lambda\omega^R_{\rho\mu,\nu} - \nabla_\lambda\omega^R_{\rho\nu,\mu} + \omega^R_{\gamma\nu,\mu}\omega^R_{\lambda\gamma,\rho} - \omega^R_{\gamma\mu,\nu}\omega^R_{\lambda\gamma,\rho} \\
& + \omega^R_{\rho\gamma,\mu}\omega^R_{\lambda\gamma,\nu} - \omega^R_{\rho\gamma,\mu}\omega^R_{\lambda\nu,\gamma} + \omega^R_{\rho\mu,\gamma}\omega^R_{\lambda\nu,\gamma} - \omega^R_{\rho\mu,\gamma}\omega^R_{\lambda\gamma,\nu} - (\lambda \leftrightarrow \rho).
\end{aligned}$$

For the scalar curvature we have

$$\begin{aligned}
R = R_{\mu\nu,\mu\nu} = {}& 2[\nabla_\mu\omega^R_{\nu\mu,\nu} - \nabla_\mu\omega^R_{\nu\nu,\mu}] + 2(\omega^R_{\mu\gamma,\mu}\omega^R_{\nu\nu,\gamma} + \omega^R_{\mu\gamma,\nu}\omega^R_{\mu\nu,\gamma} - \omega^R_{\mu\gamma,\mu}\omega^R_{\nu\gamma,\nu}) \\
& - 2(\omega^R_{\mu\gamma,\nu}\omega^R_{\mu\gamma,\nu} + \omega^R_{\mu\mu,\gamma}\omega^R_{\nu\nu,\gamma})\dagger.
\end{aligned} \tag{25}$$

Extended Cartan forms determined through equation (12) with $G = G(\phi)G(h)$ and nonzero initial conditions (14) have the form

$$\begin{aligned}
\bar{\omega}^P_\mu(\phi, h) &= [\mathrm{e}^h\,\mathrm{e}^\phi]_{\mu\nu}\,\mathrm{d}x_\nu \\
\bar{\omega}_{\mu\nu,\gamma}(\phi, h) &= [\mathrm{e}^{-h}\,\mathrm{e}^{-\phi}]_{\mu\sigma}[\mathrm{e}^{-h}\,\mathrm{e}^{-\phi}]_{\gamma\alpha}\,\partial_\alpha[\mathrm{e}^\phi\,\mathrm{e}^h]_{\sigma\nu}
\end{aligned} \tag{26}$$

where

$$[AB]_{\mu\nu} \equiv A_{\mu\sigma}B_{\sigma\nu}.$$

† The values obtained for the coefficients $c_\alpha$, $c_\beta$ may be determined from the requirement of conformal invariance of expression (21) also.

Formulae (15), (25) and (26) show that a separation of variables occurs in Cartan forms and hence in the Lagrangian.

## 6. Conclusion

The Einstein Lagrangian is expressed in terms of differential Cartan forms of the affine group by using the analogy between gravity and chiral dynamics. The each-order-invariant perturbation theory within the external background field method is formulated and is proved to be equivalent to the ordinary one on the mass shell.

Use of the covariant perturbation theory based on the Cartan forms has, as far as chiral theories are concerned, the following advantages (Kazakov *et al* 1977).

(1) Feynman diagrams within the external background field method contain a finite number of types of external line (functions of classical field) which allows one to discuss the $n$-loop approximation.

(2) Not all group invariants contribute to counterterms. Such a situation should also be expected in gravity. This question is under consideration.

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